WASHINGTON STATE UNIVERSITY

MATH 415

FINAL PROJECT

Catastrophe Theory for Musical Ensemble Dynamics

AN EXPLORATION OF FAILED PERFORMANCES AND MUSICAL IMPERFECTIONS

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Abstract

When musicians perform together in an ensemble, many factors define the tightness of the ensemble such as tempo, dynamics, intonation, and the ability for performers to adapt to changes in the performance. Musicians are constantly listening to each other and adapting their tempos to match, dynamics and timbre to create a consistent blend, and intonation to simply be in tune. Using a similar approach as Strogatz's firefly model, this paper will explore some possible models for describing the dynamics of performing musicians. A possible parameter might include a "reaction time" term which would dictate how quickly a performer is able to adapt to changes of other performers. If, for example, a performer has a very slow reaction time, small perturbations in the system might easily cause unstable oscillations and result in a catastrophic break-down of the system. We hear this in real-life performances and rehearsals when musicians get "too far off" to remain together.

I. Introduction

One of the most familiar phenomena of musical practitioners is that of failure, especially in an ensemble setting. Musicians who are playing together are constantly listening to each other and trying to match tempos, dynamics, intonation (tuning) and timbre. Each musician has to be in constant search of the balance between being influenced by the other players and setting the trends by being a leader to ensure a successful performance. However, imperfections of human performers can cause a rehearsal or, in the most unfortunate of cases, an actual performance, to come to a crashing halt. Often, this is known to happen when musicians are "getting off" from each other, viz., they somehow end up playing in different locations of the music and are unable to locate each other. Of course, most professional musicians are able to quickly locate where another performer is in the music and fix the error by jumping to the appropriate location where they can be in sync again without the audience noticing.

II. Musical Minimalism

This paper will discuss possible differential equation models and the associated system dynamics for musical ensembles. Due to the wide-varying nature of music in existence, possible scenarios involving the nature of the music (including varying tempos, technical difficulty, complexity of counterpoint, etc.), this paper will narrowly

focus on a simplistic musical model, which has real grounding in twentieth-century minimalistic music. An excellent example of which this simplified analysis will apply is Terry Riley's *In C*, which is a minimalistic piece for an arbitrary number of musicians and which consists of repeating patterns. The methodology of this composition is as follows: beginning with an initial pulse by a pianist (which is continued throughout the entire piece), the first instrument begins by playing the opening pattern. After some arbitrary amount of time, the second instrument begins playing the same pattern. When the first performer becomes bored with the pattern, he moves on to the second and the other instruments continue to follow. In this way, the music slowly evolves as the musicians progress through the slightly varying patterns. The dynamics (growing and recessive effects of volume) are defined by group movement. No single performer dictates the direction of the volume, but all move together as they "feel." The performers attempt to maintain a constant pulse throughout.

III. Exploration of Mathematical Models

This paper will model[1] a similar, limited scenario of performers playing the same melodies, taking into account dynamics where the musicians attempt to match each other's volume in addition to staying consistent with musical directions on a page. To further simplify the discussion, the case of only two musicians playing together will be discussed. This situation is very similar to Strogatz's firefly analysis where a firefly attempts to sync with a flashing bulb. Here, I will call the performers A and B. Each will have a time-instantaneous "location" in the music $x_A(t)$ and $x_B(t)$, which for simplicity, we can think of as distance[2] on the page from the start of the piece. The derivatives of these locations in the music will be the tempos: $\dot{x}_A$ and $\dot{x}_B$. Additionally, the time-instantaneous volume of each performer will be $v_A(t)$ and $v_B(t)$. Also modelled will be that the performers are attempting to follow a pre-defined "script" for the volume that is written into the music which can be represented as some arbitrary volume "signal" $v(x)$.

At the most basic level, each performer would ideally begin with the same exact musical tempo and hold steady the pulse:

$$\dot{x}_A = \omega$$

$$\dot{x}_B = \omega$$

Each musician would then be listening to the other in relation to their current location in the music and adjust accordingly. If performer A is behind performer B then performer A will increase the tempo in order to catch

[1]Disclaimer: The following mathematical model is not meant to be taken TOO seriously! It merely demonstrates the *possibility* of such modelling of musical ensembles and potential considerations.

[2]Although it would make sense for the location on the page to be musical beats, thinking of the musical location as a continuous function of time will avoid the unpleasantries of discrete-time analysis.

up. In real performances, I conjecture that the amount the tempo is increased is approximately proportional to the differences in location (to the effect that a good musician who suddenly finds herself too far behind will essentially skip portions of music to catch up). This compensation term will be some proportionality times the difference in musical locations.

$$\dot{x}_A = \omega + \kappa_A(x_B - x_A)$$

$$\dot{x}_B = \omega + \kappa_B(x_A - x_B)$$

The κ constant can be thought of as a "reaction time" constant which controls how quickly a performer will attempt to catch up or slow down.

Now we will explore the volume of the performers. The musicians, according to our simple rules defined above, will attempt to match each others volume, in addition to the volume defined on the page. In order to rectify the two conflicting sources of volume information, each performer will decide which is more important, following the instructions on the page ($v(x)$) or following the volume of the other performer ($v_{A,B}$), but take both into account. This will be achieved by a weighted average.

$$v_A = \frac{\gamma v(x_A) + \beta v_B}{\gamma + \beta}$$

$$v_B = \frac{\gamma v(x_B) + \beta v_A}{\gamma + \beta}$$

It is no surprise that performers can often have difficulty multi-tasking and that rapid changes in one area of a performance can dramatically affect the ability to control others. For example, suppose the music has technically difficult areas, a musician's attention will often be focused on "just getting through" the passage and will ignore dynamics, tempo, or other musical aspects. For our simplistic model, let's suppose that rapid dynamic changes will "distract" the performers from other musical aspects such as tempo. If the sheet music calls for the music to increase dramatically in volume very quickly, the end result will be that the performers are not able to as quickly adjust to any tempo differences to match each other. Thus, the "reaction time" constant for the tempo differences κ will become something inversely proportional to: $\dot{v}$.

$$\dot{x}_A = \omega + \frac{\kappa_A}{\dot{v}_A}(x_B - x_A)$$

$$\dot{x}_B = \omega + \frac{\kappa_B}{\dot{v}_B}(x_A - x_B)$$

Similarly, rapid fluctuations in tempo can severely limit how much a performer pays attention to volume. I will

conjecture that fluctuations in tempo have exponentially *more* of an effect on the ability to react to volume changes than does the other way around. Thus, we want the *rate* at which the volume changes to decrease if the rate of change of tempo is distracting:

$$\dot{v_A} = \frac{\gamma \dot{v}(x_A)\dot{x}_A + \beta \dot{v}_B}{\gamma + \beta} - \tau e^{\dot{x}_A}$$

$$\dot{v_B} = \frac{\gamma \dot{v}(x_B)\dot{x}_B + \beta \dot{v}_A}{\gamma + \beta} - \tau e^{\dot{x}_B}$$

Where τ controls how *much* a player's ability to adapt to changes volume is influenced by changes in tempo. A perfect player would have $\tau = 0$.

Plugging these back into our equations for musical location on the page, we have a set of nonlinear differential equations:

$$\boxed{\dot{x}_A = \omega + \frac{\kappa_A}{\left(\frac{\gamma \dot{v}(x_A)\dot{x}_A + \beta \dot{v}_B}{\gamma+\beta} - \tau e^{\dot{x}_A}\right)}(x_B - x_A)}$$

$$\boxed{\dot{x}_B = \omega + \frac{\kappa_B}{\left(\frac{\gamma \dot{v}(x_B)\dot{x}_B + \beta \dot{v}_A}{\gamma+\beta} - \tau e^{\dot{x}_B}\right)}(x_A - x_B)}$$

IV. Discussion about Analysis

Sparing the gross and gory mathematical detail, these equations could be algebraically simplified[3]. These specific equations constructed here serve the purpose of providing a rough model for the dynamics between two performers attempting to play the same musical passage while being sensitive to each other. It should be noted that these models do not take into account many details, but illustrate the possibility of constructing a set of differential equations to model musical ensembles.

Of note in these equations is the strong dependence on parameters. The reaction time constant κ gives a general sense of how apt a performer is at reacting to changes by the other performer. Although an analytical discussion will not be presented here[4], discussions of nonlinear dynamics give hint that these equations probably undergo bifurcations of some kind.

A full-scale analysis of these equations could determine if, perhaps, bifurcations occur as the parameters are varied. If so, this may in part be an explanation to the break-downs of the system we see in real life. An obvious

[3]No, I didn't get lazy. I just don't feel it is necessary to present a fully fleshed-out mathematical model. Okay...maybe some laziness did play a role. :)

[4]Of course, these specific equations are probably lacking accuracy on so many levels, it is probably not useful to perform dynamical analysis of them without any experimental confirmation, or more rigorous modelling. They are presented here only for the purpose of illustrating the *possibility* of an in-depth analysis.

simplification of this model is the assumption that the performers are "perfect" in their imperfections, meaning that reaction times are always the same, levels of distraction are constant, etc. But real performers are incredibly more complex and it would be expected that these parameters will be in constant fluctuation given other, more complex psychological, physical, and environmental factors.

V. Ruined Performance!

Catastrophe theory is the study of dynamical systems which undergo sudden, catastrophic, qualitative changes in behavior such as fixed points. While the validity and usefulness of catastrophe theory has been questioned in the past[5], some of the general ideas certainly seem applicable to musical performances. The sudden birth of limit cycles, unstable fixed points, oscillatory behavior and any other unexpected change in the dynamical system of a musical ensemble performers can easily crash a performance.

Of considerable concern is the stability of ensembles. A strong ensemble will be little affected by perturbations of system parameters, that is, restoring forces for a tight musical performance will be large. This can be clearly seen in the difference between amateur and professional musicians. Amateurs, according to the loose model presented above, have slow reaction times, and are highly "one-minded" such that they have a hard time focusing more than one aspect of the performance at a time. An ensemble of amateurs is likely to be easily interrupted or "shaky" in performance. On the other hand, professional performers have very fast reaction times and are able to have a high level of control over multiple aspects of their performance simultaneously.

Performers, amateur or professional, can improve these imperfection factors through practice. Musicians who are unfamiliar with a piece and are simply sight-reading the music, are focused on just getting the notes right and have difficulty controlling tempo or volume. But through practice, they become more familiar with the music and are able to focus on other musical aspects. A more complex mathematical model would take into account the technical difficulty of a musical passage. As the passage becomes harder to simply play the right notes, the reaction times and sensitivity to other performers would decrease. Similarly, a "familiarity" factor could be introduced which governs how familiar a musician is with the music in question. That is, how much of their mental capacity is being consumed by reading notes rather than focusing on the tightness of the ensemble?

The initial conditions of a performance can also have dramatic effect on the outcome. Many rehearsals are known to have false starts or quickly come to a crashing halt due to a bad start. If performers do not make clear the tempo they intend to start the piece, they can initially start playing the music at widely different tempos and must recognize the speed the other is taking and try to adjust. Unfamiliarity with the piece, slow reaction times, or other musical distractions can easily cause the initial tempo difference to spiral out of control.

[5]Much controversy exists over catastrophe theory. As Strogatz put it, "if you like to watch a good fight..." there are some quite contentious articles written on the subject!

VI. Conclusion

This paper presented a possible nonlinear differential set of equations to describe the simplified case of ensemble dynamics between two musicians who are playing the same music, but attempting to be musically sensitive to each other in their performance. From this model, we can easily see that there are a plethora of parameters that can be introduced to effectively model a musical ensemble. From knowledge of nonlinear dynamics, it is not hard to speculate that instabilities, bifurcations, and catastrophic "failures" of the system could easily be affected by these parameters.

It is without doubt that nonlinear dynamical analysis could shed light on these questions and perhaps even provide confirmation of intuitive aspects of a good musical ensemble or predict important aspects of performance that should be focused on to obtain a good performance. But a more thoughtful model should be constructed that is based firmly in empirical observation rather than speculation as presented here[6].

References

[1] Strogatz, Steven H. "Nonlinear Dynamics and Chaos" Harlow, England: Addison-Wesley, 1994. Paperback printing, 2000.

[2] Strogatz, Steven H. "Sync: *The Emerging Science of Spontaneous Order*." New York: Hyperion, 2003.[7]

[3] Poston, T., and Ian Stewart. "Catastrophe Theory and Its Applications". London: Pitman, 1978. [8]

[4] Kolata GB. 1977. "Catastrophe Theory: the Emperor Has No Clothes". *Science (New York, N.Y.)*. 196, no. 4287: 287-351.[9]

[6] I was unable to find any similar analyses on this subject in the manner I am proposing, so these speculations are purely my own and not founded in anything except my own experience as a music major. Hopefully the idea has some inherent merit!

[7] Strogatz wrote this general introduction to concepts of "emerging order" in science. This provides a fascinating view of how many beautiful structures and phenomena in the world work.

[8] This book from Poston and Stewart was recommended by Strogatz as an introduction to catastrophe theory.

[9] This is an almost hilarious article derailing the validity of catastrophe theory and promoting the stupidity of those who try to use it!